GENERALIZATION OF GREEN'S FORMULA

(OBOBSHCHENIE FORMULY GRINA)

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V.K. PROKOPOV
(Leningrad)

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In 1936 Lur'e introduced a symbolic method for obtaining particular solutions of the equations of elasticity for slabs [1], later this method was employed by him to obtain the differential equations of thick plates [2] and [3].

It appears tempting to try to associate the symbolic method with the principle of the minimum of potential energy, this would give the opportunity of obtaining also the natural boundary conditions. Here the variation of the potential energy will contain double integrals of the products of symbolic operators, in which the second factor is an operator acting upon the variations of the independent variables. In order to obtain the differential equations (and the boundary conditions) it is necessary to put in evidence, in these integrals, the variations of the independent variables themselves.

The transformation of the double integrals to the desired form is possible with the aid of a certain generalization of the well known Green's formula.

Employing the usual designations

$$\triangle = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}, \qquad \triangle^2 = \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)^2 = \frac{\partial^4}{\partial x^4} + 2\frac{\partial^4}{\partial x^2 \partial y^2} + \frac{\partial^4}{\partial y^4}$$

for the Laplacian raised to various exponents, let us form the infinite series

$$\Psi(\triangle) = \sum_{r=0}^{\infty} a_r \triangle^r = a_0 + a_1\triangle + a_2\triangle^2 + \ldots \tag{1}$$

in which the coefficients a_r do not depend on the independent variables x and y. This symbolic series will be referred to as an operator, in the sequel. If $\Psi(\triangle)$ is an operator of type (1), then $\Psi(\triangle)u$, where u is a function of x and y, means the following:

$$\Psi(\triangle)u = a_0 u + a_1\triangle u + a_2\triangle^2 u + \ldots$$

Let us introduce the operation of degrading the order of an operator, which consists in omitting a certain number of its first terms, with simultaneous lowering of the order of the remaining Laplacians by the number of the terms omitted. The degradation operation will be indicated by an integer subscript, the integer being equal to the number of omitted terms, that is to say the order of the degradation*. Thus, the kth degradation operator of the operator (1) is represented by the series

$$\Psi_k(\triangle) = \sum_{r=k}^{\infty} a_r \triangle^{r-k} = a_k + a_{k+1}\triangle + a_{k+2}\triangle^2 + \ldots \tag{2}$$

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* Some of the coefficients of the operator (1), including a_0 itself, may be zero.

Consider the integral

$$\iint\limits_{(\Omega)} \Phi(\triangle)\, u \cdot \Psi(\triangle)\, v\, dx\, dy, \tag{3}$$

which contains the product of operators relative to two independent functions $u(x,y)$ amd $v(x,y)$; the domain of integration is a certain domain Ω, interior to the curve L. The dot appearing between the two operators in the integral (3) denotes not only multiplication, but the complete carrying out of the operation indicated by the first operator.

Let us propose the question of transforming the two dimensional integral (3) in such a way that the Laplacians of the function $v(x,y)$ is entirely absent in the final expression. In order to carry out this objective, let us express $\Psi(\Delta)$ by means of the series (1), and interchange the operations of summation and integration, then the integral (3) becomes

$$\sum_{r=0}^{\infty} a_r \iint\limits_{(\Omega)} \Phi(\triangle)\, u \cdot \triangle^r v\, dx\, dy \tag{4}$$

Employing Green's Formula

$$\iint\limits_{(\Omega)} (\varphi \cdot \triangle\psi - \triangle\varphi \cdot \psi)\, dx\, dy = \oint\limits_{(L)} \left(\varphi \cdot \frac{\partial\psi}{\partial n} - \frac{\partial\varphi}{\partial n} \cdot \psi\right) ds$$

integrating by parts each term of the series (4), we obtain

$$\iint\limits_{(\Omega)} \Phi(\triangle)\, u \cdot \triangle^r v\, dx\, dy = \oint\limits_{(L)} \Phi(\triangle)\, u \cdot \frac{\partial}{\partial n} \triangle^{r-1} v\, ds -$$

$$- \oint\limits_{(L)} \frac{\partial}{\partial n} \Phi(\triangle)\, u \cdot \triangle^{r-1} v\, ds + \iint\limits_{(\Omega)} \triangle\Phi(\triangle)\, u \cdot \triangle^{r-1} v\, dx\, dy$$

The last term in this formula is a double integral over the domain Ω, to which one may again apply Green's formula. Repeating this operation (in all, for the rth term one has to employ Green's formula r times) we arrive at Expression

$$\iint\limits_{(\Omega)} \Phi(\triangle)\, u \cdot \triangle^r v\, dx\, dy = \sum_{p=1}^{r} \oint\limits_{(L)} \triangle^{p-1}\Phi(\triangle)\, u \cdot \frac{\partial}{\partial n} \triangle^{r-p} v\, ds -$$

$$- \sum_{p=1}^{r} \oint\limits_{(L)} \frac{\partial}{\partial n} \triangle^{p-1}\Phi(\triangle)\, u \cdot \triangle^{r-p} v\, ds + \iint\limits_{(\Omega)} \triangle^r\Phi(\triangle)\, u \cdot v\, dx\, dy \tag{5}$$

Substitution (5) into (4) gives

$$\iint\limits_{(\Omega)} \Phi(\triangle)\, u \cdot \Psi(\triangle)\, v\, dx\, dy = \sum_{r=1}^{\infty} \sum_{p=1}^{r} a_r \oint\limits_{(L)} \triangle^{p-1}\Phi(\triangle)\, u \cdot \frac{\partial}{\partial n} \triangle^{r-p} v\, ds -$$

$$- \sum_{r=1}^{\infty} \sum_{p=1}^{r} a_r \oint\limits_{(L)} \frac{\partial}{\partial n} \triangle^{p-1}\Phi(\triangle)\, u \cdot \triangle^{r-p} v\, ds + \sum_{r=0}^{\infty} a_r \iint\limits_{(\Omega)} \triangle^r\Phi(\triangle)\, u \cdot v\, dx\, dy \tag{6}$$

Consider the first double sum in Formula (6). Interchanging the order of summation, summing first over r and then over $k = r - p + 1$ (instead of summing first over p and then over r, as it stands). The orders of the Laplacians become $p - 1 = r - k$ and $r - p = k - 1$. The inner summation (with respect to r) is then extended from k to ∞, (as follows from Formula $r = k + p - 1$, since the smallest $p = 1$), the outer summation (with respect to k) is extended from 1 to ∞. Then, recalling the definition of an operator (2), we obtain

$$\sum_{k=1}^{\infty} \sum_{r=k}^{\infty} \oint\limits_{(L)} a_r \triangle^{r-k}\Phi(\triangle)\, u \cdot \frac{\partial}{\partial n} \triangle^{k-1} v\, ds = \sum_{k=1}^{\infty} \oint\limits_{(L)} \Psi_k(\triangle)\, \Phi(\triangle)\, u \cdot \frac{\partial}{\partial n} \triangle^{k-1} v\, ds$$

An analogue transformation may be applied to the second double sum in Equation (6). The last sum, after interchanging the order of summation and integration, gives the operator $\Psi(\Delta)$ under the integral sign. In view of what has been said, we obtain following final Formula of the transformation of the integral (3)

$$\iint\limits_{(\Omega)} \Phi(\Delta)\, u\cdot\Psi(\Delta)\, v\, dx\, dy = \iint\limits_{(\Omega)} \Psi(\Delta)\,\Phi(\Delta)\, u\cdot v\, dx\, dy +$$
$$+ \sum_{k=1}^{\infty} \oint\limits_{(L)} \left[\Psi_k(\Delta)\,\Phi(\Delta)\, u\cdot \frac{\partial \Delta^{k-1} v}{\partial n} - \frac{\partial}{\partial n}\Psi_k(\Delta)\,\Phi(\Delta)\, u\cdot\Delta^{k-1} v\right] ds \tag{7}$$

This Formula represents a generalization of Green's formula to the case of infinite operators. Formula (7) also remains valid in three-dimensional case. Here the double and curvilinear integrals are replaced by triple and surface integrals respectively, and the Laplace operator becomes three-dimensional

$$\iiint\limits_{(V)} \Phi(\Delta)\, u\cdot\Psi(\Delta)\, v\, dV = \iiint\limits_{(V)} \Psi(\Delta)\,\Phi(\Delta)\, u\cdot v\, dV + \qquad \left(\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right)$$
$$+ \sum_{k=1}^{\infty} \iint\limits_{(S)} \left[\Psi_k(\Delta)\,\Phi(\Delta)\, u\cdot \frac{\partial \Delta^{k-1} v}{\partial n} - \frac{\partial}{\partial n}\Psi_k(\Delta)\,\Phi(\Delta)\, u\cdot\Delta^{k-1} v\right] ds \tag{8}$$

In the one-dimensional case, together with the operator

$$\Psi(\partial) = \sum_{n=0}^{\infty} a_n \partial^n \qquad \left(\Delta = \frac{\partial^2}{\partial x^2} = \partial^2\right) \tag{9}$$

one has to consider its adjoint operator

$$\Psi^*(\partial) = \sum_{n=0}^{\infty} (-1)^n a_n \partial^n \tag{10}$$

The definite integral

$$\int_a^b \Phi(\partial)\, u\cdot\Psi(\partial)\, v\, dx = \sum_{n=0}^{\infty} a_n \int_a^b \Phi(\partial)\, u\cdot\partial^n v\, dx \tag{11}$$

may be transformed by integration by parts. The coefficient of a_n in the series (11) we have

$$\int_a^b \Phi(\partial)\, u\cdot\partial^n v\, dx = \left[\sum_{p=1}^{n} (-1)^{p-1}\partial^{p-1}\Phi(\partial)\, u\cdot\partial^{n-p} v\right]_{x=a}^{x=b} + (-1)^n \int_a^b \partial^n\Phi(\partial)\, u\cdot v\, dx \tag{12}$$

Substitution of (12) into series (11) yields

$$\int_a^b \Phi(\partial)\, u\cdot\Psi(\partial)\, v\, dx = \left[\sum_{n=1}^{\infty}\sum_{p=1}^{n} (-1)^{p-1} a_n \partial^{p-1}\Phi(\partial)\, u\cdot\partial^{n-p} v\right]_{x=a}^{x=b} +$$
$$+ \sum_{n=0}^{\infty} (-1)^n a_n \int_a^b \partial^n\Phi(\partial)\, u\cdot v\, dx \tag{13}$$

In the double sum appearing in Formula (13), let us interchange the order of summation; let us sum first over n, and then over $k = n - p + 1$. Then the inner summation (with respect to n) will proceed from k to ∞, while the outer summation (with respect to k) is taken from 1 to ∞. Then one obtains

$$\int_a^b \Phi(\partial)\, u \cdot \Psi(\partial)\, v\, dx = \left[\sum_{k=1}^{\infty} \sum_{n=k}^{\infty} (-1)^{n-k} a_n \partial^{n-k} \Phi(\partial)\, u \cdot \partial^{k-1} v\right]_{x=a}^{x=b} + \sum_{n=0}^{\infty} (-1)^n a_n \int_a^b \partial^n \Phi(\partial)\, u \cdot v\, dx \qquad (14)$$

The combination of Formulas (2) and (10) gives the degraded conjugate operator

$$\Psi_k^*(\partial) = \sum_{n=k}^{\infty} (-1)^n a_n \partial^{n-k} \qquad (15)$$

Taking (10) and (15) into account, we obtain from Formula (14) final Equation

$$\int_a^b \Phi(\partial)\, u \cdot \Psi(\partial)\, v\, dx = \left[\sum_{k=1}^{\infty} (-1)^k \Psi_k^*(\partial)\, \Phi(\partial)\, u \cdot \partial^{k-1} v\right]_{x=a}^{x=b} + \int_a^b \Psi^*(\partial)\, \Phi(\partial)\, u \cdot v\, dx \qquad (16)$$

which is a generalization of the formula for integration by parts in the case of one-dimensional operators of infinite order.

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